TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 329, Number 1, January 1992

THE GREEN CORRESPONDENCE FOR THE REPRESENTATIONS OF HECKE ALGEBRAS OF TYPE A_{r-1}

JIE DU

ABSTRACT. We first prove the conjecture mentioned by Leonard K. Jones in his thesis. By applying this conjecture, we obtain that the vertex of an indecomposable $\mathscr{H}_F$-module is an l-parabolic subgroup. Finally, we establish the Green correspondence for the representations of Hecke algebras of type A_{r-1}.

INTRODUCTION

Let R be a $\mathbf{Q}[u^{1/2}]$-algebra in which $u^{1/2}$ is invertible. Let (W, S) be the symmetric group on r letters where S is the set of basic transpositions. Then the Hecke algebra $\mathscr{H}_R$ corresponding to W is a free R-module with basis $\{\widetilde{T}_w ; w \in W\}$ which obey the following multiplication rules (see [Du]):

$$\widetilde{T}_w \widetilde{T}_s = \begin{cases} \widetilde{T}_{ws}, & \text{if } w < ws, \\ (u^{-1/2} - u^{1/2})\widetilde{T}_w + \widetilde{T}_{ws}, & \text{otherwise}, \end{cases}$$

where "$<$" is the Bruhat order and $w \in W$, $s \in S$.

The study of the representations of the Hecke algebra $\mathscr{H}_R$ has turned out many remarkable q-analogues of the representations of the symmetric groups (see [DJ1 and 2, Ho and Jo]). In this paper we shall generalize some basic results of Green along the lines of the work of L. Jones (see [Jo]). We organize the paper as follows: After recalling some basic results, we shall prove the conjecture (Theorem 2.7) which has been mentioned in [Jo, 5.3]. The Brauer homomorphism constructed by Jones will play a key role in proving that conjecture. With the aid of this conjecture, we obtain that the vertex of an indecomposable $\mathscr{H}_R$ module is an l-parabolic subgroup. In the last section, we shall establish the Green correspondence for the representations of Hecke algebra $\mathscr{H}_R$.

I would like to thank Professor Leonard Scott for his valuable ideas and comments concerning that conjecture. I also wish to thank the University of Virginia for its hospitality during the writing of this paper.

Received by the editors July 5, 1989 and, in revised form, November 5, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 16A64; Secondary 20C20.
Key words and phrases. Hecke algebra, relative norm, relatively projective, vertex, Green correspondence.
The author was supported in part by the N.S.F.

1. Induced and indecomposable modules

Let l be a positive integer, $l \leq r$, and $\Phi_l(u^{1/2})$ is the lth cyclotomic polynomial in $u^{1/2}$. Let R_l be the completion of the polynomial ring in the indeterminate $u^{1/2}$ over $\mathbf{Q}$ localized at the maximal ideal generated by $\Phi_l(u^{1/2})$. Let K be the quotient field of R_l and F the residue class field $R_l/\eta R_l$ where η is the generator of the maximal ideal of R_l.

We call that (K, R_l, F) is a characteristic 0 modular system. Let $R \in \{K, R_l, F\}$.

Let λ be a composition of r. (A composition λ of r, denoted $\lambda \vDash r$, is a finite sequence $(\lambda_1, \lambda_2, \dots, \lambda_n)$ of nonnegative integers whose sum is r.) Then the standard Young (or the parabolic) subgroups W_λ of W consists of those permutations of $\{1, 2, \dots, r\}$ which leave invariant the following sets of integers $\{1, 2, \dots, \lambda_1\}$, $\{\lambda_1+1, \lambda_1+2, \dots, \lambda_1+\lambda_2\}$, $\{\lambda_1+\lambda_2+1, \dots\}, \dots$. A parabolic subgroup H is called l-parabolic if, for each proper parabolic subgroup H' of H,

$$\Phi_l(u) | \frac{d_H}{d_{H'}}$$

where $d_{W'}$ denotes the Poincaré polynomial of W' for any parabolic subgroup W' of W.

If W' is a parabolic subgroup of W, we denote by $\mathscr{D}_{W'}$ the set of all distinguished coset representatives of right cosets of W' in W and set $\mathscr{D}_\lambda = \mathscr{D}_{W'}$ if $W' = W_\lambda$. Let $\mathscr{D}_{\lambda\mu} = \mathscr{D}_\lambda \cap \mathscr{D}_\mu^{-1}$. Then $\mathscr{D}_{\lambda\mu}$ is the set of distinguished $W_\lambda - W_\mu$ double coset representatives. Also, the R-module $\sum_{w \in W'} RT_w$ is a subalgebra of $\mathscr{H}_R$, which is called the parabolic subalgebra of $\mathscr{H}_R$, denoted by $\mathscr{H}_{W'}$. We will use the abbreviation $\mathscr{H}_\lambda$ instead of $\mathscr{H}_{W_\lambda}$.

If M is an $\mathscr{H}_R$-module and N is an $\mathscr{H}_\lambda$-module, then we denote by $M_{\mathscr{H}_\lambda}$ the restriction of M from $\mathscr{H}_R$-module and denote by $N^{\mathscr{H}_R} = N \otimes_{\mathscr{H}_\lambda} \mathscr{H}_R$ the induced module of N. The q-analogue of Mackey's decomposition theorem for finite groups holds (see [DJ1, Jo]).

1.1 **Theorem** (Mackey's decomposition). *Let $\lambda, \mu \vDash r$ and let N be an $\mathscr{H}_R$-$\mathscr{H}_\lambda$ bimodule then*

$$(N^{\mathscr{H}_R})_{\mathscr{H}_\mu} \cong \sum_{d \in \mathscr{D}_{\lambda\mu}} [(N \otimes_{\mathscr{H}_\lambda} \widetilde{T}_d) \otimes_{\mathscr{H}_{\nu(d)}} \mathscr{H}_\mu],$$

where $\nu(d)$ is defined by $W_{\nu(d)} = W_\lambda^d \cap W_\mu$ for all $d \in \mathscr{D}_{\lambda\mu}$. □

Let M be a finitely generated indecomposable right $\mathscr{H}_R$-module. Then, by [Jo, 3.35], there exists a parabolic subgroup W_λ of W unique up to conjugation such that M is relatively $\mathscr{H}_\lambda$-projective (see the definition in [Jo, Chapter 3]) and such that W_λ is W-conjugate to a parabolic subgroup of any parabolic subgroup W_μ of W for which M is relatively $\mathscr{H}_\mu$-projective.

We call W_λ the *vertex* of M. In the next section we shall see that the vertex of M must be an l-parabolic subgroup of W.

Mackey's theorem can be used to prove the following result.

1.2 **Proposition.** *Let W_τ be the vertex of the indecomposable $\mathscr{H}_R$-module M. Then*

(a) *There is an indecomposable $\mathscr{H}_\tau$-module N such that $M | N^{\mathscr{H}_R}$.*

(b) *If* N' *is another indecomposable* $\mathscr{H}_\tau$*-module with the property* (a), *then there is an element*

$$d \in N_W(W_\tau) \cap \mathscr{D}_{\tau\tau}$$

such that

$$N \cong N' \otimes_{\mathscr{H}_\tau} \widetilde{T}_d$$

as $\mathscr{H}_\tau$*-modules.*

The notation $X|Y$ means that X is isomorphic to a direct summand of Y and $N_W(W_\tau)$ denotes the normaliser of W_τ in W.

Proof. Since M is relatively $\mathscr{H}_\tau$-projective, we have

$$M|M \otimes_{\mathscr{H}_\tau} \mathscr{H}_R .$$

Thus, there is an indecomposable direct summand N of $M_{\mathscr{H}_\tau}$ such that

$$M|N \otimes_{\mathscr{H}_\tau} \mathscr{H}_R .$$

Hence (a) follows.

Assume that V is an indecomposable $\mathscr{H}_\tau$-module with

$$M|V \otimes_{\mathscr{H}_\tau} \mathscr{H}_R .$$

Since $N|M_{\mathscr{H}_\tau}$ we have

$$N|(V \otimes_{\mathscr{H}_\tau} \mathscr{H}_R)_{\mathscr{H}_\tau} .$$

By Mackey's theorem,

$$(V \otimes_{\mathscr{H}_\tau} \mathscr{H}_R)_{\mathscr{H}_\tau} \cong \bigoplus_{d \in \mathscr{D}_{\tau\tau}} (V \otimes_{\mathscr{H}_\tau} \widetilde{T}_d \otimes_{\mathscr{H}_{\nu(d)}} \mathscr{H}_\tau)$$

where $\nu(d)$ is defined by $W_{\nu(d)} = W_\tau^d \cap W_\tau$. Thus by Krull-Schmidt theorem, there is $d \in \mathscr{D}_{\tau\tau}$ such that

$$N|V \otimes_{\mathscr{H}_\tau} \widetilde{T}_d \otimes_{\mathscr{H}_{\nu(d)}} \mathscr{H}_\tau .$$

Therefore

$$M|V \otimes_{\mathscr{H}_\tau} \widetilde{T}_d \otimes_{\mathscr{H}_{\nu(d)}} \mathscr{H}_R .$$

Since W_τ is the vertex of M it follows from Higman's criterion [Jo, 3.34] that

$$W_\tau \subseteq_W W_{\nu(d)} = W_\tau^d \cap W_\tau .$$

Therefore

$$W_\tau = W_\tau^d \cap W_\tau \quad \text{and} \quad d \in N_W(W_\tau) .$$

Thus we have

$$N|V \otimes_{\mathscr{H}_\tau} \widetilde{T}_d .$$

Since V is indecomposable, so is $V \otimes_{\mathscr{H}_\tau} \widetilde{T}_d$, hence

$$N \cong V \otimes_{\mathscr{H}_\tau} \widetilde{T}_d$$

as $\mathscr{H}_\tau$-modules. □

The module N is called a *source* of M.

To study the representations of Hecke algebras, the relative norm plays a remarkable role (see [Du, Jo]). Here is the definition of relative norms.

1.3 **Definition.** Let λ, μ be compositions of r such that $W_\lambda \subseteq W_\mu$. Let M be an $\mathscr{H}_\mu$-$\mathscr{H}_\mu$ bimodule and $b \in M$. Define the *relative norm*

$$N_{W_\mu, W_\lambda}(b) = \sum_{w \in \mathscr{D}_\lambda \cap W_\mu} \widetilde{T}_{w^{-1}} b \widetilde{T}_w .$$

There are a number of nice properties of relative norms which will be used freely in the subsequent discussion. These results are mostly due to P. Hoefsmit and L. Scott. One can find a complete proof in [Jo, Chapter 3].

Let M be an $\mathscr{H}_\lambda$-$\mathscr{H}_\lambda$ bimodule. We define

$$Z_M(\mathscr{H}_\lambda) = \{m \in M \mid hm = mh \text{ for all } h \in \mathscr{H}_\lambda\} .$$

Obviously $Z(\mathscr{H}_\lambda) = Z_{\mathscr{H}_\lambda}(\mathscr{H}_\lambda)$ is the center of $\mathscr{H}_\lambda$ and, for $M = \mathrm{Hom}_R(N, N)$ where N is a right $\mathscr{H}_\lambda$-module,

$$Z_M(\mathscr{H}_\lambda) = \mathrm{Hom}_{\mathscr{H}_\lambda}(N, N) .$$

One can describe a basis of the center $Z(\mathscr{H}_R)$ of $\mathscr{H}_R$ or a basis of the q-Schur algebra $S_R(n, r)$ in terms of relative norms (see [Du, Jo]), which give us much more facilities.

Let $P_k = W_{(l^k, 1^{r-kl})}$ where k satisfies $r = kl + s$, $s < l$. In [Jo, 5.3] there is a conjecture as follows:

1.4 *Conjecture.* $N_{W, P_k}(1)$ is invertible.

We shall prove this conjecture in the next section.

2. The invertibility of $N_{W, P_k}(1)$

In [Jo] a Brauer-type homomorphism is constructed and the image of certain basis of $Z(\mathscr{H}_R)$ is discussed modulo a conjecture which has been proved in [Sh, Proposition 11]. The proof of Proposition 2.6 below is motivated by the argument in [Jo].

Let $\gamma = (l, r-l)$. Then $\mathscr{H}_\gamma \cong \mathscr{H}_l \otimes \mathscr{H}_{r-1}$ where $\mathscr{H}_l = \mathscr{H}_{(l, 1^{r-l})}$ and $\mathscr{H}_{r-l} = \mathscr{H}_{(1^l, r-l)}$. Since W is a disjoint union of the subsets W_γ and $W'xW'$ for $x \notin W_\gamma$ and $W' = W_{(l, 1^{r-l})}$, that is,

$$W = W_\gamma \cup \left(\bigcup_{x \notin W_\gamma} W'xW' \right)$$

(see the proof of [Jo, 5.1.5]), we have

$$Z(\mathscr{H}_R) \subseteq Z_{\mathscr{H}_R}(\mathscr{H}_\gamma) = Z(\mathscr{H}_\gamma) \oplus Z_{M_r}(\mathscr{H}_\gamma) \tag{2.a}$$

where $M_r = \bigoplus_{x \notin W_\gamma} \mathscr{H}_l \widetilde{T}_x \mathscr{H}_l$.

Let π be the projection of $Z(\mathscr{H}_R)$ onto $Z(\mathscr{H}_\gamma)$ and let σ be the canonical map from $Z(\mathscr{H}_\gamma)$ onto $Z(\mathscr{H}_\gamma)/N_{W',1}(\mathscr{H}_\gamma) \cap Z(\mathscr{H}_\gamma)$. Then the map

$$f = \sigma \circ \pi \colon Z(\mathscr{H}_R) \to Z(\mathscr{H}_\gamma)/N_{W',1}(\mathscr{H}_\gamma) \cap Z(\mathscr{H}_\gamma)$$

is an algebraic homomorphism which is called the Brauer homomorphism, following Jones.

2.1 **Lemma.** *Let* π *be as above and let*

$$h = \sum_w a_w \widetilde{T}_w \in Z_{\mathscr{H}_R}(\mathscr{H}_\gamma).$$

Then $\pi(h) = 0$ *if and only if* $s_l = (l, l+1) < w$ *for all* w *with* $a_w \neq 0$.

Proof. Immediate from the fact that $w \notin W_\gamma$ if and only if $s_l < w$. □

2.2 **Lemma.** $Z(\mathscr{H}_\gamma)/N_{W',1}(\mathscr{H}_\gamma) \cap Z(\mathscr{H}_\gamma) \cong [Z(\mathscr{H}_l)/N_{W',1}(\mathscr{H}_l)] \otimes Z(\mathscr{H}_{r-l})$.

Proof. We first claim that

$$N_{W',1}(\mathscr{H}_l) \otimes Z(\mathscr{H}_{r-l}) = (N_{W',1}(\mathscr{H}_l) \otimes \mathscr{H}_{r-l}) \cap (Z(\mathscr{H}_l) \otimes Z(\mathscr{H}_{r-l})). \tag{2.b}$$

Obviously, the left-hand side of (2.b) is contained in the right-hand side. Let $\{v_i,\ 1 \le i \le s\}$ be a basis of $Z(\mathscr{H}_l)$ such that $\{v_i,\ 1 \le i \le t\}$ is a basis of $N_{W',1}(\mathscr{H}_l)$. If $a = \sum_{i=1}^t v_i \otimes a_i$ is an element of right-hand side of (2.b) then it is easy to see $a_i \in Z(\mathscr{H}_{r-l})$. Hence the claim is proved. Thus by the claim,

$$\begin{aligned} &Z(\mathscr{H}_\gamma)/N_{W',1}(\mathscr{H}_\gamma) \cap Z(\mathscr{H}_\gamma) \\ &\quad\cong [Z(\mathscr{H}_l) \otimes Z(\mathscr{H}_{r-l})]/[N_{W',1}(\mathscr{H}_l) \otimes \mathscr{H}_{r-l} \cap Z(\mathscr{H}_l) \otimes Z(\mathscr{H}_{r-l})] \\ &\quad\cong [Z(\mathscr{H}_l) \otimes Z(\mathscr{H}_{r-l})]/[N_{W',1}(\mathscr{H}_l) \otimes Z(\mathscr{H}_{r-l})] \\ &\quad\cong [Z(\mathscr{H}_l)/N_{W',1}(\mathscr{H}_l)] \otimes Z(\mathscr{H}_{r-l}). \end{aligned}$$

Hence the result. □

Let $Z = Z(\mathscr{H}_l)/N_{W',1}(\mathscr{H}_l)$ and let $r = kl + s$, $s < l$. For each m, $1 \le m \le k$ we define

$$f_m = (\underbrace{\mathrm{id}_Z \otimes \cdots \otimes \mathrm{id}_Z}_{m-1} \otimes f) \circ \cdots \circ (\mathrm{id}_Z \otimes f) \circ f.$$

Then f_m is a homomorphism from $Z(\mathscr{H}_R)$ into

$$\underbrace{Z \otimes \cdots \otimes Z}_{m} \otimes Z(\mathscr{H}_{r-ml}).$$

If e is a central primitive idempotent of $\mathscr{H}_R$ then we say that the *defect* of e is m if $f_m(e) \neq 0$ and $f_{m+1}(e) = 0$.

2.3 **Lemma.** *Let* e *be a central primitive idempotent of* $\mathscr{H}_R$. *Then* $eZ(\mathscr{H}_R)$ *is a local ring.*

Proof. Let

$$\mathscr{A} = \mathscr{H}_R^{op} \otimes \mathscr{H}_R.$$

Then $\mathscr{H}_R$ is an $\mathscr{A}$-module defined by

$$h(h_1 \otimes h_2) = h_1 h h_2$$

for $h, h_1, h_2 \in \mathscr{H}_R$, and the $\mathscr{A}$-submodules of $\mathscr{H}_R$ are ideals of $\mathscr{H}_R$. So $e\mathscr{H}_R$ is an indecomposable $\mathscr{A}$-submodule. Hence $\mathrm{End}_{\mathscr{A}}(e\mathscr{H}_R)$ is a local ring.

Since $\mathrm{End}_{\mathscr{H}_R}(e\mathscr{H}_R) = e\mathscr{H}_R$, it follows that

$$\mathrm{End}_{\mathscr{A}}(e\mathscr{H}_R) = Z(e\mathscr{H}_R) = eZ(\mathscr{H}_R),$$

hence the result. □

2.4 **Lemma.** *Let* e *be a central primitive idempotent and* $f(e) \neq 0$. *Then there exist pairwise orthogonal primitive idempotents* $\{e_{i1}\}_{1\le i\le s}$, $\{e_{i2}\}_{1\le i\le s}$ *of* Z, $Z(\mathscr{H}_{r-l})$ *respectively such that*

$$f(e) = \sum_{i=1}^{s} e_{i1} \otimes e_{i2}.$$

Moreover, if the defect of e *is* m *then the defect of* $e_{i2} \le m-1$ *for all* i.

Proof. Let $\{e_i\}$, $\{e'_j\}$ be the pairwise orthogonal primitive idempotents of Z, $Z(\mathscr{H}_{r-l})$ respectively such that

$$1_Z = \sum_i e_i, \quad 1_{Z(\mathscr{H}_{r-l})} = \sum_j e'_j.$$

Then

$$1_{Z\otimes Z(\mathscr{H}_{r-l})} = \sum_{i,j} e_i \otimes e'_j$$

is a decomposition of primitive idempotents of the identity of $Z \otimes Z(\mathscr{H}_{r-l})$. Since $f(e)$ is an idempotent of $Z \otimes Z(\mathscr{H}_{r-l})$ we may find an expression of $f(e)$ as desired.

Suppose that there is t, $1 \le t \le s$, such that e_{t2} is of defect $\ge m$. Thus $f_m(e_{t2}) \neq 0$ and therefore

$$f_{m+1}(e) = \sum_{i=1}^{s} \mathrm{id}_Z(e_{i1}) \otimes f_m(e_{i2}) \neq 0.$$

This is contrary to our assumption. So the defect of $e_{i2} \le m-1$ for all i. □

Let $\imath$ denote the anti-automorphism of $\mathscr{H}_R$ defined by $\imath(\widetilde{T}_w) = \widetilde{T}_{w^{-1}}$ and $\imath^2 = 1$. Let $P_m = W_{(1^{(k-m)l}, l^m, 1^{r-kl})}$, $0 \le m \le k$.

2.5 **Lemma.** *Let* e_0 *be a central primitive idempotent of* $\mathscr{H}_R$ *with defect* 0. *Then* $N_{W,P_k}(1)e_0$ *is invertible in* $e_0 Z(\mathscr{H}_R)$.

Proof. Let χ be the irreducible character of $\mathscr{H}_R$ over $R = \mathbf{Q}(u)$ associated with $e' = \imath(e_0)$. d_χ denotes the generic degree of χ. Then

$$\frac{d_W}{d_\chi} \not\equiv 0 \mod(\Phi_l)$$

(see [Jo, 5.2.29]). By the proof of [Jo, 3.5] we have

$$N_{W,1}(1)e' = \frac{d_W}{d_\chi}\chi(1)e' \not\equiv 0 \mod(\Phi_l).$$

Therefore,

$$\mathrm{Tr}(N_{W,1}(1)e') = \frac{d_W}{d_\chi}\chi(1)^2 \not\equiv 0 \mod(\Phi_l).$$

On the other hand, we have

$$\begin{aligned}
\mathrm{Tr}(N_{W,1}(1)e') &= \mathrm{Tr}(N_{W,P_k}(N_{P_k,1}(1))e') \\
&= \sum_{w\in\mathscr{D}_{P_k}} \mathrm{Tr}(\widetilde{T}_{w^{-1}} N_{P_k,1}(1)\widetilde{T}_w e') \\
&= \sum_{w\in\mathscr{D}_{P_k}} \mathrm{Tr}(\widetilde{T}_w \widetilde{T}_{w^{-1}} e' N_{P_k,1}(1)) \\
&= \mathrm{Tr}(\imath(N_{W,P_k}(1))e' N_{P_k,1}(1)).
\end{aligned}$$

Since $\iota(N_{W,P_k}(1))e'$ commutes with $N_{P_k,1}(1)$ we have $\iota(N_{W,P_k}(1)e_0)$ is not nilpotent, hence $N_{W,P_k}(1)e_0$ is not nilpotent. Hence $N_{W,P_k}(1)e_0$ is invertible in $e_0Z(\mathscr{H}_R)$ since $e_0Z(\mathscr{H}_R)$ is a local ring. □

We now fix some notation. Let

$$d_0 = 1\,, \quad d_m = (l\,, l+m)(l-1\,, l+m-1)\cdots(l-m+1\,, l+1)$$

for $1 \le m \le m(l) = \min\{l\,, r-l\}$. Then by [Jo, (5.2.2)]

$$\mathscr{D}_{\gamma\gamma} = \{d_m \mid 0 \le m \le m(l)\}$$

and

$$G_m = W_\gamma^{d_m} \cap W_\gamma = W_{(l-m\,, m\,, m\,, r-l-m)}\,.$$

2.6 **Proposition.** *Let F be the Brauer homomorphism,*

$$f\colon Z(\mathscr{H}_R) \to [Z(\mathscr{H}_l)/N_{W',1}(\mathscr{H}_l)] \otimes Z(\mathscr{H}_{r-l})\,.$$

Then

$$f(N_{W,P_k}(1)) = kN_{W_\gamma,P_k}(1)\,.$$

Proof. By the transitivity of relative norm we have

$$\begin{aligned} N_{W,P_k}(1) &= N_{W,W_\gamma}(N_{W_\gamma,P_k}(1)) \\ &= \sum_{m=0}^{m(l)} N_{W_\gamma,G_m}(\widetilde{T}_{d_m} N_{W_\gamma,P_k}(1)\widetilde{T}_{d_m})\,. \end{aligned} \tag{2.c}$$

Let $W'' = W_{(1^l,r-l)}$. Then $G_m = G_{m1} \times G_{m2}$ where $G_{m1}\,, G_{m2}$ are the intersections of G_m with $W'\,, W''$ respectively. Since

$$N_{W_\gamma,P_k}(1) = N_{W'',P_{k-1}}(1) \in Z_{\mathscr{H}_{W''}}(\mathscr{H}_{G_{m2}})$$

we may write by (2.a)

$$N_{W_\gamma,P_k}(1) = N_m + T_m$$

where $N_m \in Z(\mathscr{H}_{G_{m2}})$ and

$$T_m = \sum_{\substack{z \in W'' \\ z \notin G_{m2}}} b_z \widetilde{T}_z\,.$$

Thus, if $d_m z\, d_m \in W_\gamma$ with $\widetilde{T}_z$ involved in T_m, then

$$d_m z\, d_m \in W_\gamma^{d_m} \cap W_\gamma = G_m$$

hence

$$z \in G_m^{d_m} = G_m\,.$$

It follows that $(l+m\,, l+m+1) \in G_m$ since $z \in W''$ and $z \notin G_{m2}$. This is impossible. Therefore $d_m z\, d_m \notin W_\gamma$. Thus each term $\widetilde{T}_w$ involved in $\widetilde{T}_{d_m}\widetilde{T}_z\widetilde{T}_{d_m}$ satisfies $s_l < w$. Therefore, by 2.1,

$$\pi(N_{W_\gamma,G_m}(\widetilde{T}_{d_m} T_m \widetilde{T}_{d_m})) = 0\,.$$

We now examine N_m. By [Jo, 4.33] we express N_m as a linear combination of the basis of $Z(\mathscr{H}_{G_{m2}})$:

$$N_m = \sum_{\substack{\alpha \vdash (r-l) \\ W_\alpha \subseteq G_{m2}}} a_\alpha N_{G_{m2},W_\alpha}(\eta_\alpha)$$

where $\eta_\alpha \in Z(\mathscr{H}_{W_\alpha})$.

Let

$$\eta_\alpha = \sum_{w \in W_\alpha} a_w \widetilde{T}_w .$$

Then by [Jo, 5.2.21] we have

$$\begin{aligned}
\widetilde{T}_{d_m}\eta_\alpha \widetilde{T}_{d_m} &= \sum_{w\in G_{m2}} a_w \widetilde{T}_{d_m}\widetilde{T}_w\widetilde{T}_{d_m} \\
&= \sum_{w\in G_{m2}} a_w \left(\widetilde{T}_{d_m w d_m} + \sum_{\substack{x\in W\\ s_l<x}} b_{wx}\widetilde{T}_x\right) \\
&= \eta_{d_m\alpha\, d_m} + \sum_{s_l<z} c_z \widetilde{T}_z
\end{aligned}$$

where $\eta_{d_m\alpha\, d_m} \in Z(\mathscr{H}_{W_\alpha^{d_m}})$. Thus if we denote $S_\alpha = G_{m1} \times W_\alpha$ then by [Jo, 5.2.10] we have

$$\begin{aligned}
\widetilde{T}_{d_m} N_{G_{m2}, W_\alpha}(\eta_\alpha)\widetilde{T}_{d_m} &= \widetilde{T}_{d_m} N_{G_m, S_\alpha}(\eta_\alpha)\widetilde{T}_{d_m} \\
&= \sum_{x\in\mathscr{D}_{S_\alpha}\cap G_m} \widetilde{T}_{d_m x^{-1}}\eta_\alpha \widetilde{T}_{x\, d_m} \\
&= \sum_{\hat{x}\in\mathscr{D}_{S_\alpha^{d_m}}\cap G_m} \widetilde{T}_{\hat{x}^{-1}}(\widetilde{T}_{d_m}\eta_\alpha\widetilde{T}_{d_m})\widetilde{T}_{\hat{x}} \\
&= N_{G_m, S_\alpha^{d_m}}(\eta_{d_m\alpha\, d_m}) + \sum_{s_l<w} d_w \widetilde{T}_w .
\end{aligned}$$

Therefore, by (2.1),

$$\pi\left(N_{W_\gamma, G_m}\left(\sum_{s_l<w} d_w \widetilde{T}_w\right)\right) = 0$$

and if $0 < m < l$ then $N_{G_{m1}^{d_m}, 1}(1)$ is invertible by [Jo], thus

$$\begin{aligned}
N_{W_\gamma, G_m}(N_{G_m, S_\alpha^{d_m}}(\eta_{d_m\alpha\, d_m})) &= N_{W', G_{m1}^{d_m}}(1) N_{W'', W_\alpha^{d_m}}(\eta_{d_m\alpha d_m}) \\
&= N_{W', 1}\left(\frac{1}{N_{G_{m1}^{d_m}, 1}(1)} N_{W'', W_\alpha^{d_m}}(\eta_{d_m\alpha\, d_m})\right)
\end{aligned}$$

which lies in the kernel of σ.

By observing (2.c) and the above arguments we obtain that

$$f(N_{W, P_k}(1)) = \begin{cases} N_{W_\gamma, P_k}(1), & \text{if } m(l) < l, \\ \sum_{m=0, l} f(N_{W_\gamma, G_m}(\widetilde{T}_{d_m} N_{W_\gamma, P_k}(1)\widetilde{T}_{d_m})), & \text{if } m(l) = l. \end{cases}$$

In particular, if $k = 1$ then $m(l) < l$ and

$$f(N_{W, P_k}(1)) = N_{W_\gamma, P_k}(1).$$

So the assertion is true for $k = 1$. Assume now that $k > 1$. Then $m(l) = l$ and we have

$$f(N_{W, P_k}(1)) = N_{W_\gamma, P_k}(1) + f(N_{W_\gamma, G_l}(\widetilde{T}_{d_l} N_{W_\gamma, P_k}(1)\widetilde{T}_{d_l})).$$

By induction we get

$$N_{W_\gamma, P_k}(1) = N_{W'', P_{k-1}}(1) = (k-1)N_{W_{\gamma'}, P_{k-1}}(1) + \sum_{\substack{w\in W'' \\ (2l, 2l+1)<w}} f_w \widetilde{T}_w$$

where $W_{\gamma'} = W_{(1^l, l, r-2l)}$. Thus similar reason as before shows that

$$\pi\left(N_{W_\gamma, G_l}\left(\widetilde{T}_{d_l} \sum_{\substack{w\in W'' \\ (2l, 2l+1)<w}} f_w \widetilde{T}_w \widetilde{T}_{d_l}\right)\right) = 0.$$

So we eventually obtain that

$$\begin{aligned} f(N_{W, P_k}(1)) &= N_{W_\gamma, P_k}(1) + f(N_{W_\gamma, G_l}(\widetilde{T}_{d_l}(k-1)N_{W_{\gamma'}, P_{k-1}}(1)\widetilde{T}_{d_l})) \\ &= N_{W_\gamma, P_k}(1) + (k-1)f(N_{W_\gamma, G_l}(\widetilde{T}_{d_l}N_{G_l, P_k}(1)\widetilde{T}_{d_l})) \\ &= N_{W_\gamma, P_k}(1) + (k-1)f(N_{W_\gamma, P_k}(\widetilde{T}_{d_l}^2)) \\ &= N_{W_\gamma, P_k}(1) + (k-1)N_{W_\gamma, P_k}(1) \\ &= kN_{W_\gamma, P_k}(1) \end{aligned}$$

since $P_k^{d_l} = P_k$ and

$$\widetilde{T}_{d_l}^2 = \widetilde{T}_1 + \sum_{\substack{w\in W \\ s_l<w}} g_w \widetilde{T}_w$$

by [Jo, 5.2.20]. □

2.7 **Theorem.** *Let e be a central primitive idempotent of $\mathscr{H}_R$. Then $N_{W, P_k}(1)e$ is invertible in $eZ(\mathscr{H}_R)$. Therefore $N_{W, P_k}(1)$ is invertible in $\mathscr{H}_R$.*

Proof. The first statement is true if e is of defect 0 by (2.5). Assume that e is of defect $m > 0$. By (2.4)

$$f(e) = \sum_{i=1}^{s} e_{i1} \otimes e_{i2}$$

where e_{i1}, e_{i2} are the central primitive idempotents of Z, $Z(\mathscr{H}_{r-l})$ respectively, and the defect of $e_{i2} < m$. Thus by the previous proposition,

$$\begin{aligned} f(N_{W, P_k}(1)e) &= f(N_{W, P_k}(1))f(e) \\ &= kN_{W_\gamma, P_k}(1)\sum_i e_{i1} \otimes e_{i2} \\ &= k\sum_i e_{i1} \otimes N_{W'', P_{k-1}}(1)e_{i2}. \end{aligned}$$

By induction we have that $N_{W'', P_{k-1}}(1)e_{i2}$ is invertible in $e_{i2}Z(\mathscr{H}_{r-l})$ for all i, and $\{e_{i1} \otimes e_{i2}\}_{1\le i\le s}$ is orthogonal pairwise. So $f(N_{W, P_k}(1)e)$ is not nilpotent. Therefore, $N_{W, P_k}(1)e$ is not nilpotent. So it is invertible in $eZ(\mathscr{H}_R)$ since $eZ(\mathscr{H}_R)$ is a local ring.

Let $1 = \sum_{i=1}^{s} e_i$ be a decomposition of central primitive idempotents of the identity of $\mathscr{H}_R$. Then

$$N_{W, P_k}(1) = \sum_{i=1}^{s} N_{W, P_k}(1)e_i.$$

Since $N_{W,P_k}(1)e_i$ is invertible in $e_iZ(\mathscr{H}_R)$ for all i we have $N_{W,P_k}(1)$ is invertible. □

3. Green correspondence

In this section we shall present a couple of applications of Theorem 2.7.

Recall from §1 that if M is an indecomposable $\mathscr{H}_R$-module then there is a "minimal" parabolic subgroup W_λ of W such that M is relatively $\mathscr{H}_\lambda$-projective. Such a group W_λ is called a vertex of M. Now we can say more about the vertex.

3.1 **Theorem.** *Let M be a finitely generated indecomposable $\mathscr{H}_R$-module. Then the vertex of M is an l-parabolic subgroup of W.*

Proof. Let H be the vertex of M and P the maximal l-parabolic subgroup of H. Then, by Higman's criterion,

$$N_{W,H}(\mathrm{Hom}_{\mathscr{H}_H}(M, M)) = \mathrm{Hom}_{\mathscr{H}_R}(M, M).$$

Since $N_{H,P}(1)$ is invertible, we have

$$\begin{aligned} N_{W,H}(\mathrm{Hom}_{\mathscr{H}_H}(M, M)) &= N_{W,P}\left(\frac{1}{N_{H,P}(1)}\mathrm{Hom}_{\mathscr{H}_H}(M, M)\right) \\ &\subseteq N_{W,P}(\mathrm{Hom}_{\mathscr{H}_P}(M, M)) \end{aligned}$$

therefore

$$N_{W,P}(\mathrm{Hom}_{\mathscr{H}_P}(M, M)) = \mathrm{Hom}_{\mathscr{H}_R}(M, M).$$

By Higman's criterion again we get M is relatively $\mathscr{H}_P$-projective. This forces $P = H$. □

In the modular representation theory of finite groups, the Green correspondence connects indecomposable modules for the group G with modules for its local subgroups (see [Al, F]). Now, we start to establish a q-analogue of the Green correspondence for the representations of Hecke algebras. Such a generalization becomes apparent as soon as 3.1 is set up. First of all we need a couple of lemmas.

3.2 **Lemma.** *Let M be an indecomposable $\mathscr{H}_R$-module with vertex W_τ and W_ρ is a parabolic subgroup containing W_τ. Then there is an indecomposable $\mathscr{H}_\rho$-module N satisfying any two of the following statements*:

(a) $N | M_{\mathscr{H}_\rho}$;
(b) $M | N \otimes_{\mathscr{H}_\rho} \mathscr{H}_R$;
(c) *N has vertex W_τ.*

Proof. Since W_τ is a vertex of M we have

$$M | M \otimes_{\mathscr{H}_\tau} \mathscr{H}_R.$$

Thus,

$$M | (M \otimes_{\mathscr{H}_\tau} \mathscr{H}_\rho) \otimes_{\mathscr{H}_\rho} \mathscr{H}_R$$

since $W_\tau \subseteq W_\rho$. Hence, M is relatively $\mathscr{H}_\rho$-projective by Higman's criterion. Therefore

$$M | M \otimes_{\mathscr{H}_\rho} \mathscr{H}_R.$$

Thus, there is an indecomposable summand N of $M_{\mathscr{H}_\rho}$ such that

$$M|N\otimes_{\mathscr{H}_\rho}\mathscr{H}_R.$$

So (a) and (b) hold.

Let V be a source of M. Then $M|V\otimes_{\mathscr{H}_\tau}\mathscr{H}_R$ and hence, $M|(V^{\mathscr{H}_\rho})^{\mathscr{H}_R}$. Thus there is an indecomposable summand N of $V^{\mathscr{H}_\rho}$ with $M|N^{\mathscr{H}_R}$. We claim that N has vertex W_τ.

Since $N|V^{\mathscr{H}_\rho}$, we have N is relatively $\mathscr{H}_\tau$-projective, so there is a vertex $W_{\tau'}$ of N with $W_{\tau'}\subseteq W_\tau$. Let V' be a $\mathscr{H}_{\tau'}$-module with $N|(V')^{\mathscr{H}_\rho}$. Then

$$N^{\mathscr{H}_R}|((V')^{\mathscr{H}_\rho})^{\mathscr{H}_R}\quad\text{and}\quad((V')^{\mathscr{H}_\rho})^{\mathscr{H}_R}=(V')^{\mathscr{H}_R},$$

hence

$$M|N^{\mathscr{H}_R}|V'\otimes_{\mathscr{H}_{\tau'}}\mathscr{H}_R.$$

That is M is relatively $\mathscr{H}_{\tau'}$-projective. Thus $W_{\tau'}$ contains a conjugate of W_τ. Since $W_{\tau'}\subseteq W_\tau$ we have $W_{\tau'}=W_\tau$. (b) and (c) are true.

By the proof of 1.2 there is an indecomposable $\mathscr{H}_\tau$-module V such that $V|M_{\mathscr{H}_\tau}$ and $M|V^{\mathscr{H}_R}$. Hence there is an indecomposable $\mathscr{H}_\rho$-module N with $N|M_{\mathscr{H}_\rho}$ and $V|N_{\mathscr{H}_\tau}$. We shall prove that N has vertex W_τ.

Since $N|M_{\mathscr{H}_\rho}$ we have $N|(V^{\mathscr{H}_R})_{\mathscr{H}_\rho}$. By Mackey's theorem there exists $d\in\mathscr{D}_{\tau\rho}$ such that

$$N|V\otimes_{\mathscr{H}_\tau}\widetilde{T}_d\otimes_{\mathscr{H}_{\nu(d)}}\mathscr{H}_\rho.$$

Hence N has a vertex $W_{\tau'}$ with $W_{\tau'}\subseteq W_\tau^d\cap W_\rho=W_{\nu(d)}$. Assume that V' is a source of N, $N|V'\otimes_{\mathscr{H}_{\tau'}}\mathscr{H}_\rho$. Thus

$$V|(V'\otimes_{\mathscr{H}_{\tau'}}\mathscr{H}_\rho)_{\mathscr{H}_\tau}.$$

By Mackey's theorem we see that V is relatively $\mathscr{H}_{W_{\tau'}^z\cap W_\tau}$-projective for some z. Thus M is also relatively $\mathscr{H}_{W_{\tau'}^z\cap W_\tau}$-projective, hence $W_\tau\subseteq_W W_{\tau'}^z\cap W_\tau$. Therefore,

$$W_{\tau'}=W_\tau^d\qquad(d\in\mathscr{D}_{\tau\rho})$$

since $W_{\tau'}\subseteq W_\tau^d$. Hence W_τ is a vertex of N. (a) and (c) hold. □

3.3 **Lemma.** *Let τ, ρ be as in 3.2. If N is a relatively $\mathscr{H}_\tau$-projective $\mathscr{H}_\rho$-module, then*

$$(N\otimes_{\mathscr{H}_\rho}\mathscr{H}_R)_{\mathscr{H}_\rho}\cong N\oplus Y$$

where every indecomposable summand of Y is relatively projective for a subgroup of the form

$$W_\tau^d\cap W_\rho,\quad\textit{for } d\in\mathscr{D}_{\tau\rho},\ d\neq 1.$$

Proof. Since N is $\mathscr{H}_\tau$-projective we have $N|V\otimes_{\mathscr{H}_\tau}\mathscr{H}_\rho$ for some $\mathscr{H}_\tau$-module V. Thus

$$V\otimes_{\mathscr{H}_\tau}\mathscr{H}_\rho=N\oplus T$$

for some $\mathscr{H}_\rho$-module T.

Now,

$$V\otimes_{\mathscr{H}_\tau}\mathscr{H}_R\cong(N\otimes_{\mathscr{H}_\rho}\mathscr{H}_R)\oplus(T\otimes_{\mathscr{H}_\rho}\mathscr{H}_R)$$

and

$$(V\otimes_{\mathscr{H}_\tau}\mathscr{H}_R)_{\mathscr{H}_\rho}\cong N\oplus Y\oplus T\oplus X$$

where

$$(N \otimes_{\mathscr{H}_\rho} \mathscr{H}_R)_{\mathscr{H}_\rho} = N \oplus Y\,, \quad (T \otimes_{\mathscr{H}_\rho} \mathscr{H}_R)_{\mathscr{H}_\rho} = T \oplus X$$

for suitable $\mathscr{H}_\rho$-modules X, Y by Mackey's theorem.

On the other hand, by Mackey's theorem again,

$$(V \otimes_{\mathscr{H}_\tau} \mathscr{H}_R)_{\mathscr{H}_\rho} = \bigoplus_{d \in \mathscr{D}_{\tau\rho}} (V \otimes_{\mathscr{H}_\tau} \widetilde{T}_d \otimes_{\mathscr{H}_{\nu(d)}} \mathscr{H}_\rho) = V \otimes_{\mathscr{H}_\tau} \mathscr{H}_\rho \oplus U$$

where $d = 1$ gives $V \otimes_{\mathscr{H}_\tau} \mathscr{H}_\rho$, the U is the direct sum of all terms for $d \neq 1$ and so each indecomposable summand of U is relatively projective for a subgroup of the form $W_\tau^d \cap W_\rho$, $d \in \mathscr{D}_{\tau\rho}$, $d \neq 1$. The Krull-Schmidt theorem implies that $X \oplus Y \cong U$. So Y is as claimed. □

From now on we assume that $W_\rho = W_{(ml\,,\,r-ml)}$, $W_\theta = W_{(l^m\,,\,1^{r-ml})}$. Then

$$W_\rho \supseteq N_W(W_\theta) \supseteq W_{(l^m\,,\,r-ml)} \supseteq W_\theta\,.$$

Let $\mathscr{P}$ be the collection of all parabolic subgroups of W. If $\mathscr{S}$ is a collection of parabolic subgroups of W, then $P \in_W \mathscr{S}$ for $P \in \mathscr{P}$ means $P = H^x$ for some $H \in \mathscr{S}$, $x \in W$.

We say that $\mathscr{H}_R$-module M is relatively $\mathscr{S}$-projective if $M = \bigoplus_i M_i$, and each M_i is relatively projective for a group in $\mathscr{S}$.

Let

$$\begin{aligned} \mathscr{X} &= \{H \in \mathscr{P} | H \subseteq W_\theta^d \cap W_\theta \text{ for some } d \in W\,,\, d \notin W_\rho\}\,, \\ \mathscr{Y} &= \{H \in \mathscr{P} | H \subseteq W_\theta^d \cap W_\rho \text{ for some } d \in W\,,\, d \notin W_\rho\}\,, \\ \mathscr{Z} &= \{P \in \mathscr{P} | P \subseteq W_\theta \text{ is } l\text{-parabolic, } P \notin_W \mathscr{X}\}\,. \end{aligned}$$

Observe that $W_\rho \supseteq N_W(W_\theta)$, $\mathscr{X}$ consists of proper subgroups of W_θ, but $W_\theta \in \mathscr{Z}$.

3.4 **Lemma.** *If W_τ is an l-parabolic subgroup of W_θ then the following assertions are equivalent*:

(a) $W_\tau \in_W \mathscr{X}$;
(b) $W_\tau \in \mathscr{X}$;
(c) $W_\tau \in \mathscr{Y}$;
(d) $W_\tau \in_{W_\rho} \mathscr{Y}$.

Proof. (a) ⇒ (b) If (a) holds then there exists $x \in W$ with

$$W_\tau \subseteq (W_\theta^d \cap W_\theta)^x$$

for some $d \in W$, $d \notin W_\rho$. Since either dx or x is not in W_ρ we have $W_\tau \in \mathscr{X}$.

(b) ⇒ (c) Obvious, since $\mathscr{X} \subseteq \mathscr{Y}$.

(c) ⇒ (d) Obvious.

(d) ⇒ (a) Suppose that (d) holds. Then there exist $x \in W_\rho$, $d \in W$, $d \notin W_\rho$ with

$$W_\tau \subseteq (W_\theta^d \cap W_\rho)^x\,.$$

Thus,

$$W_\tau \subseteq W_\theta^{dx} \cap W_\theta$$

and $dx \notin W_\rho$. Hence, $W_\tau \in_W \mathscr{X}$. □

3.5 **Corollary.** (a) *If M is relatively $\mathscr{X}$-projective $\mathscr{H}_R$-module then $M_{\mathscr{H}_\rho}$ is $\mathscr{Y}$-projective;*

(b) *If N is relatively $\mathscr{Y}$-projective $\mathscr{H}_\rho$-module with a vertex contained in W_θ for each indecomposable summand of N, then $N^{\mathscr{H}_R}$ is relatively $\mathscr{X}$-projective.*

Proof. If L is an indecomposable summand of M then L is relatively projective for a parabolic subgroup W_λ of the group $W_\theta^d \cap W_\theta$, for $d \in W$, $d \notin W_\rho$. Thus we have

$$L | L \otimes_{\mathscr{H}_\lambda} \mathscr{H}_R .$$

By Mackey's theorem, $L_{\mathscr{H}_\rho}$ is relatively projective for the collection $\mathscr{S}$,

$$\mathscr{S} = \{Q \in \mathscr{P} | Q \subseteq W_\lambda^z \cap W_\rho \text{ for some } z \in \mathscr{D}_{\lambda\rho}\}.$$

Since $W_\lambda^z \cap W_\rho \subseteq (W_\theta^d \cap W_\theta)^z \cap W_\rho$ and either dz or z is not in W_ρ we have $\mathscr{S} \subseteq \mathscr{Y}$. Therefore $L_{\mathscr{H}_\rho}$ and $M_{\mathscr{H}_\rho}$ are relatively $\mathscr{Y}$-projective.

On the other hand, if L is an indecomposable summand of N and L has a vertex W_τ with $W_\tau \in \mathscr{Y}$, $W_\tau \subseteq W_\theta$, then $L^{\mathscr{H}_R}$ is relatively $\mathscr{H}_\tau$-projective and $W_\tau \in \mathscr{X}$ by 3.4. So $L^{\mathscr{H}_R}$ and $N^{\mathscr{H}_R}$ is relatively $\mathscr{X}$-projective. □

We now prove our main result in this section, which is a q-analogue of the Green correspondence in the representation theory of finite groups.

3.6 **Theorem.** *There is a one to one correspondence between isomorphic classes of indecomposable $\mathscr{H}_R$-modules with vertex in $\mathscr{Z}$ and isomorphic classes of indecomposable $\mathscr{H}_\rho$-modules with vertex in $\mathscr{Z}$, which can be characterized as follows:*

(a) *Let M be an indecomposable $\mathscr{H}_R$-module with vertex W_τ in $\mathscr{Z}$. Then $M_{\mathscr{H}_\rho}$ has a unique indecomposable direct summand $f(M)$ with W_τ as vertex. Furthermore,*

$$M_{\mathscr{H}_\rho} \cong f(M) \oplus \left(\bigoplus_i N_i\right)$$

where a vertex N_i lies in $\mathscr{Y}$ for all i.

(b) *Let N be an indecomposable $\mathscr{H}_\rho$-module with vertex W_τ in $\mathscr{Z}$. Then $N^{\mathscr{H}_R}$ has a unique indecomposable direct summand $g(N)$ with W_τ as vertex. Furthermore,*

$$N^{\mathscr{H}_R} = g(N) \oplus \left(\bigoplus_j M_j\right)$$

where M_j has a vertex in $\mathscr{X}$ for all j.

(c) *In particular, $g(f(M)) = M$ and $f(g(N)) = N$.*

Proof. (a) By 3.2 there is an indecomposable $\mathscr{H}_\rho$-module V with vertex W_τ and

$$M | V \otimes_{\mathscr{H}_\rho} \mathscr{H}_R . \tag{3.a}$$

Applying 3.3 we obtain

$$(V \otimes_{\mathscr{H}_\rho} \mathscr{H}_R)_{\mathscr{H}_\rho} = V \oplus Y_1$$

where Y_1 is $\mathscr{Y}$-projective. Thus, $M_{\mathscr{H}_\rho}$ is isomorphic to $V \oplus Y$ or Y for some summand Y of Y_1. However, again by 3.2, $M_{\mathscr{H}_\rho}$ has an indecomposable summand V' with vertex W_τ. Now we claim that V' cannot be isomorphic to a summand of Y_1. Otherwise, $W_\tau \in_{W_p} \mathscr{Y}$ by [Jo, 3.35], hence $W_\tau \in_W \mathscr{X}$ by 3.4. This is contrary to $W_\tau \in \mathscr{Z}$. Hence $V' \cong V$ and

$$(3.b) \qquad M_{\mathscr{H}_p} \cong V \oplus Y$$

just as claimed. The argument also shows V is unique up to isomorphism. Let $f(M) = V$. Then f is well defined.

(b) Let

$$(3.c) \qquad N^{\mathscr{H}_R} = M_1 \oplus M_2 \oplus \cdots \oplus M_t$$

be a direct sum of indecomposable $\mathscr{H}_R$-module. Since, by 3.3,

$$(N^{\mathscr{H}_R})_{\mathscr{H}_p} \cong N \oplus Y$$

where Y is relatively $\mathscr{Y}$-projective, we have, after renumbering, that

$$(3.d) \qquad (M_1)_{\mathscr{H}_p} \cong N \oplus Y_1, \quad (M_i)_{\mathscr{H}_p} \cong Y_i, \qquad 2 \le i \le t,$$

where the Y_i's are $\mathscr{H}_p$-modules and

$$Y \cong Y_1 \oplus Y_2 \oplus \cdots \oplus Y_t.$$

We claim that M_1 has a vertex in $\mathscr{Z}$ and that $M_2, \ldots, M_t$ are $\mathscr{X}$-projective.

Indeed, since $M_i | N \otimes_{\mathscr{H}_p} \mathscr{H}_R$ and $N | N \otimes_{\mathscr{H}_\tau} \mathscr{H}_p$ we have

$$M_i | N \otimes_{\mathscr{H}_\tau} \mathscr{H}_R.$$

Hence M_i has a vertex contained in W_θ. Let $W_{\tau'} \subseteq W_\theta$ be a vertex of M_1. Suppose that M_1 is relatively $\mathscr{X}$-projective. Then 3.5 implies that $(M_1)_{\mathscr{H}_p} \cong N \oplus Y_1$ is relatively $\mathscr{Y}$-projective. It follows that $W_\tau \in_{W_p} \mathscr{Y}$ since the vertex of N is W_τ, hence $W_\tau \in_W \mathscr{X}$ by 3.4, a contradiction. So M_1 is not $\mathscr{X}$-projective, that is $W_{\tau'} \notin_W \mathscr{X}$ by 3.4 again. Hence $W_{\tau'} \in \mathscr{Z}$, as claimed.

Moreover, if M_i, $(i > 1)$ was not relatively $\mathscr{X}$-projective then, by 3.5, $(M_i)_{\mathscr{H}_p}$ would not be relatively $\mathscr{Y}$-projective since M_i has a vertex contained in W_θ, a contradiction. Hence M_i $(i > 1)$ is indeed relatively $\mathscr{X}$-projective. Let $g(N) = M_1$. We have seen that $g(N)$ is unique up to isomorphism, so g is well defined.

It remains to check (c). By (3.a) and (3.c) we have

$$M | f(M)^{\mathscr{H}_R}, \quad f(M)^{\mathscr{H}_R} \cong g(f(M)) \oplus X$$

where X is relatively $\mathscr{X}$-projective. Hence, $g(f(M)) \cong M$. Similarly, (3.b) and (3.d) imply $f(g(N)) \cong N$.

References

[Al] J. L. Alperin, *Local representation theory*, Cambridge Univ. Press, 1986.

[Du] Jie Du, *The modular representation theory of q-Schur algebras*, Trans. Amer. Math. Soc. **329** (1992), 253–271.

[DJ1] R. Dipper and G. James, *Representations of Hecke algebras of general linear groups*, Proc. London Math. Soc. **52** (1986), 20–52.

[DJ2] ——, *Blocks and idempotents of Hecke algebras of general linear groups*, Proc. London Math. Soc. **53** (1987), 57–82.

[DJ3] ——, *The q-Schur algebras*, Proc. London Math. Soc. **59** (1989), 23–50.

[DJ4] ——, *q-tensor spaces and q-Weyl modules*, Trans. Amer. Math. Soc. **327** (1991), 251–282.

[F] W. Feit, *The representation theory of finite groups*, North-Holland Mathematical Library, vol. 25, North-Holland, 1982.

[G] J. A. Green, *Blocks of modular representations*, Math. Z. **79** (1962), 100–115.

[G1] ——, *Polynomial representations of GL_n*, Lecture Notes in Math., vol. 830, Springer-Verlag, Berlin and New York, 1980.

[Ho] P. Hoefsmit, *Representations of Hecke algebras of finite groups with (B, N)-pairs of classical type*, Ph.D. Dissertation, Univ. of British Columbia, Vancouver, 1974.

[Jo] L. Jones, *Centers of generic algebras*, Ph.D. Thesis, Univ. of Virginia, 1987.

[S] L. Scott, *Modular permutation representations*, Trans. Amer. Math. Soc. **175** (1973), 101–121.

[Sh] Jianyi Shi, *A result on the Bruhat order of a Coxeter group*, J. Algebra (to appear).

Department of Mathematics, University of Virginia, Charlottesville, Virginia 22903-3199

Permanent address: Department of Mathematics, East China Normal University, Shanghai 200062, China

Current address: Department of Pure Mathematics, University of Sydney, N.S.W. 2006, Australia